Stochastic Stokes' drift

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February 2, 2008

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Abstract

Classical Stokes' drift is the small time-averaged drift velocity of suspended non-diffusing particles in a fluid due to the presence of a wave. We consider the effect of adding diffusion to the motion of the particles, and show in particular that a non-zero time-averaged drift velocity exists in general even when the classical Stokes' drift is zero. Our results are obtained from a general procedure for calculating ensemble-averaged Lagrangian mean velocities for motion that is close to Brownian, and are verified by numerical simulations in the case of sinusoidal forcing.

PACS numbers: 02.50-r, 05.40+j, 05.60+w

A travelling wave in a fluid gives suspended particles a small drift velocity known as Stokes' drift [1, 2, 3]. When there is more than one wave, the drift velocity is calculated by summing the contributions from each wave [4, 5]. In this letter we consider the influence of diffusion on the magnitude and direction of the drift velocity. As in the classical (diffusion-less) case, the amplitude of the travelling wave is assumed small compared to its wavelength;

a non-zero drift velocity appears at second order in the amplitude. In the presence of more than one wave, the classical Stokes' drift can sum to zero. Diffusion then produces a non-zero drift velocity whose magnitude and direction depends on the diffusivity of the suspended particles.

Several mechanisms for the directed motion of small particles without a net macroscopic force have been proposed in the last 10 years [6, 7, 8, 9, 10]. Interest in such 'ratchet' effects has been motivated by the search for the mechanisms of biological motors, such as the conversion of chemical energy into directed motion by protein molecules, and by possible applications, such as the separation of particles in solution based on their diffusion coefficients. In both these cases small particles are believed to follow dynamics that are overdamped (first derivative in time) and noise-dominated. A drift velocity dependent on the size of suspended particles in solution has been produced experimentally using an asymmetric periodic potential turned on and off periodically [11]. Published theoretical models [12, 13] combine a periodic asymmetric potential in one dimension with non-white fluctuations.

In this letter we consider motion in arbitrary dimensions that is diffusion-dominated. There is also a small deterministic forcing whose amplitude will be used as an expansion parameter; a drift velocity appears at second order and depends on the diffusivity. Thus diffusion due to microscopic motions, for example diffusion of particles in solution, can be exploited using a carefully-chosen combination of forcings to produce a net motion that depends on the diffusivity. We illustrate the effect with sinusoidal forcing and compare our calculations with numerical results in one and two space dimensions. It is possible to arrange the wave motions so that particles of different diffusivities have a time-averaged drift velocity in different directions, resulting in what we call 'fan-out'. This may have applications for sorting particles according to their diffusivities. We show numerically that the fan-out can have an angular range of more than 180 degrees.

We first develop an expansion scheme for motion that is overdamped and diffusion-dominated. Consider a stochastic process $\mathbf{X} \equiv (\mathbf{X}_t)_{t\geq 0}$ taking values in $\mathbb{R}^m$ and satisfying

the following stochastic differential equation [14, 15]:

$$\mathrm{d}\mathbf{X}_t = \epsilon f(\mathbf{X}_t, t)\mathrm{d}t + \mathrm{d}\mathbf{W}_t, \quad 0 \leq \epsilon \ll 1. \tag{1}$$

The vector $\mathbf{X}_t$ is the particle position at time t. Its ensemble average, to be denoted below by angled brackets, is the Lagrangian mean position at time t. $\mathbf{W}$ is an m-dimensional Brownian motion, with $\mathbf{W}_0 = 0$ and $\langle \mathbf{W}_t \cdot \mathbf{W}_t \rangle = m\sigma^2 t$, i.e. $\mathbf{W}$ represents a purely diffusive motion, with diffusivity

$$D = \tfrac{1}{2}\sigma^2. \tag{2}$$

The remaining term in (1) is the deterministic forcing, a function of Eulerian position x and time t:

$$f : \mathbb{R}^m \times \mathbb{R}^+ \to \mathbb{R}^m. \tag{3}$$

The real constant ϵ satisfies $0 \leq \epsilon \ll 1$.

We now expand in powers of ϵ. Let

$$\mathbf{X}_t = \mathbf{X}_t^{(0)} + \epsilon \mathbf{X}_t^{(1)} + \epsilon^2 \mathbf{X}_t^{(2)} + \ldots, \quad \text{with initial condition } \ \mathbf{X}_0 = 0. \tag{4}$$

The leading terms of the stochastic equation of motion, equation (4), are as follows.

ϵ^0 :

$$\mathrm{d}\mathbf{X}_t^{(0)} = \mathrm{d}\mathbf{W}_t, \tag{5}$$

giving

$$\mathbf{X}_t^{(0)} = \mathbf{W}_t. \tag{6}$$

ϵ^1 :

$$\mathrm{d}\mathbf{X}_t^{(1)} = f(\mathbf{X}_t^{(0)}, t)\mathrm{d}t, \tag{7}$$

giving

$$\mathbf{X}_t^{(1)} = \int_0^t f(\mathbf{W}_s, s)\,\mathrm{d}s. \tag{8}$$

ϵ^2 :

$$\mathrm{d}\mathbf{X}_t^{(2)} = (\mathbf{X}_t^{(1)} \cdot \nabla) f(\mathbf{X}_t^{(0)}, t)\mathrm{d}t, \tag{9}$$

giving the second-order drift velocity as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}_t^{(2)} = \int_0^t (f(\mathbf{W}_s, s) \cdot \nabla) f(\mathbf{W}_t, t)\, \mathrm{d}s, \tag{10}$$

where ∇f is the spatial gradient of f.

In the classical derivation of Stokes' drift there is no motion at zeroth order [1, 2]. Here the motion at zeroth order is purely diffusive, with $\langle \mathbf{X}_t^{(0)} \rangle = 0$ for any positive time. When $f(x, t)$ is a sum of functions that are periodic in t at any fixed x, the drift velocity also vanishes at first order in ϵ. That is

$$\lim_{t\to\infty} \frac{1}{t} \langle \mathbf{X}_t^{(1)} \rangle = 0. \tag{11}$$

At second-order the drift velocity is in general non-zero. It is given by the following ensemble average:

$$\begin{aligned} U &\equiv \epsilon^2 \lim_{t\to\infty} \frac{1}{t} \langle \mathbf{X}_t{}^{(2)} \rangle & (12) \\ &= \epsilon^2 \lim_{T\to\infty} \left(\frac{1}{T} \int_0^T \Big\langle \int_0^t \left(f(\mathbf{W}_s, s) \cdot \nabla \right) f(\mathbf{W}_t, t)\, \mathrm{d}s \Big\rangle \mathrm{d}t \right). & (13) \end{aligned}$$

In one space dimension the expression (13) reduces to

$$U \equiv \epsilon^2 \lim_{T\to\infty} \left(\frac{1}{T} \int_0^T \Big\langle f'(\mathbf{W}_t, t) \int_0^t f(\mathbf{W}_s, s)\, \mathrm{d}s \Big\rangle\, \mathrm{d}t \right), \tag{14}$$

where $f'(x, t) = \frac{\partial}{\partial x} f(x, t)$.

We now consider the case where the deterministic forcing is a sum of sinusoids:

$$f(x, t) = \sum_{\ell=1}^{n} A_\ell k_\ell \cos(k_\ell \cdot x - \omega_\ell t), \tag{15}$$

where A_ℓ and ω_ℓ are constants. The vector k_ℓ defines the direction of propagation of wave l. The drift velocity (13) for this case is

$$U \;=\; \epsilon^2 \sum_{\ell=1}^{n} \Big[\lim_{t\to\infty} A_\ell^2 |k_\ell|^2 k_\ell \int_0^t \Big\langle \sin\big(- k_\ell \cdot (\mathbf{W}_t - \mathbf{W}_s) + \omega_\ell (t - s)) \big) \Big\rangle \mathrm{d}s \Big]$$

$$= \tfrac{1}{2}\epsilon^2 \sum_{\ell=1}^{n} \Big[\lim_{t\to\infty} A_\ell^2 |k_\ell|^2 k_\ell \int_0^t \sin(\omega_\ell(t-s))\mathrm{e}^{-|k_\ell|^2 D(t-s)}\mathrm{d}s\Big] \tag{16}$$

$$= \tfrac{1}{2}\epsilon^2 \sum_{\ell=1}^{n} \left[A_\ell^2 |k_\ell|^2 \frac{k_\ell}{\omega_\ell}\left(1 + D^2 \frac{|k_\ell|^4}{\omega_\ell^2}\right)^{-1}\right].$$

Each wave makes a contribution to the drift velocity in its direction of propagation. For $D = 0$, the weighting factor is proportional to the square of the amplitude. This is the classical result obtained by a transformation from Eulerian to Lagrangian coordinates [1, 2]. In the case of a surface wave over deep water, the first order motion of a suspended particle is a circle with radius $A_\ell k_\ell/\omega_\ell$; the quantity $A_\ell^2|k_\ell|^2 k_\ell/\omega_\ell$ is proportional to the time-averaged momentum per unit area [2, 3]. In the presence of diffusion, the contribution from wave ℓ is reduced by the dimensionless factor $(1+\alpha_\ell^2)^{-1}$, where $\alpha_\ell = D|k_\ell|^2\omega_\ell^{-1}$. Diffusion reduces the Stokes drift due to any one wave by smearing out the distribution of particles, working against the tendency of particles to spend longer in regions where the force acts in the direction of propagation than in those where the force acts in the opposite direction. The attenuation is strongest for waves with large wavenumbers or small velocities.

Dependence of drift velocity on diffusion can be exploited as follows: there is in general a non-zero drift velocity due to diffusion even when the classical Stokes' drift is zero. We write the drift velocity (16) as a sum of the classical Stokes' drift and a diffusion-dependent contribution:

$$U = U_0 + U_\mathrm{s}, \tag{17}$$

where

$$U_0 = U|_{D=0} = \tfrac{1}{2}\epsilon^2 \sum_{\ell=1}^{n} A_\ell^2 |k_\ell|^2 \frac{k_\ell}{\omega_\ell} \tag{18}$$

and

$$U_\mathrm{s} = -\tfrac{1}{2}\epsilon^2 \sum_{\ell=1}^{n} \left[A_\ell^2 |k_\ell|^2 \frac{k_\ell}{\omega_\ell}\frac{\alpha_\ell^2}{1+\alpha_\ell^2}\right]. \tag{19}$$

The classical Stokes drift U_0 can be made to vanish by choosing a forcing $f(x,t)$ consisting of two wave trains propagating in opposite directions. For the latter example, we can work in one space dimension, defined as the direction of propagation of wave $\ell = 1$:

$$f(x,t) = A_1 k_1 \cos(k_1 x - \omega_1 t + \phi_1) + A_2 k_2 \cos(k_2 x - \omega_2 t + \phi_2), \tag{20}$$

where A_i, k_i, ω_i and ϕ_i $(i = 1, 2)$ are constants and $k_1 k_2 < 0$. For simplicity, we suppose that $k_1 \neq \pm k_2$ and $\omega_1 \neq \pm\omega_2$; this avoids cross-terms in the classical Stokes' drift. The drift velocity including diffusion is then given by

$$U = \tfrac{1}{2}\epsilon^2 \left(A_1^2 \frac{k_1^3}{\omega_1} \left(1 + \alpha_1^2\right)^{-1} + A_2^2 \frac{k_2^3}{\omega_2} \left(1 + \alpha_2^2\right)^{-1} \right). \tag{21}$$

To set $U_0 = 0$ requires $A_1^2 k_1^3/\omega_1 = -A_2^2 k_2^3/\omega_2$. Then $U = U_{\mathrm{s}}$ where

$$U_{\mathrm{s}} = \tfrac{1}{2}\epsilon^2 A_1^2 \frac{k_1^3}{\omega_1} \left(\left(1 + \alpha_1^2\right)^{-1} - \left(1 + \alpha_2^2\right)^{-1} \right). \tag{22}$$

For large diffusivity the drift velocity tends to zero because the contribution of each wave tends to zero. Thus there is an intermediate value of diffusivity that maximises U_{s}. If the forcing frequencies and wavenumbers are fixed and $U_0 = 0$, the drift attains its maximum at the value of D satisfying $\alpha_1\alpha_2 = 1$.

Figure 1 shows the drift velocity as a function of diffusivity with the forcing a sum of two sinusoids for a choice of parameters that gives $U_0 = 0$. In Figure 2 the calculated drift is compared with numerical results, with the same choice of parameters and $D = 0.125$. The solid line in Figure 2(a) is the mean value of $\mathbf{X}_t$ as a function of time, averaged over 10000 numerical realizations of the stochastic differential equation (1), and the dotted line is Ut, with U given by (22). In Figure 2(b) we show, as a function of time, the difference between the numerically-calculated mean displacement and Ut. Figure 2(c) demonstrates that the motion is close to Brownian; a histogram of values of $\mathbf{X}_t$ at $t = 1000$, $R(y)$, is compared with the Gaussian probability density function with mean Ut and variance $\sigma^2 t$ (solid line).

In general the expressions (16) and (19) are vector relations. Thus, in more than one space dimension, the direction as well as the magnitude of the drift velocity depends on the diffusivity, producing fan-out of the drift velocity vectors. We illustrate this effect in Figure 3, constructed with the forcing being a sum of four sinusoids in two dimensions. In (a), the vector $A_\ell k_\ell$ is shown for each of the four waves. The parameters are $A_1 = 1.0$, $A_2 = 0.8$, $A_3 = 0.7$, $A_4 = 0.7$; $k_1 = (1.0, 0.0)$, $k_2 = (2.0, -4.0)$, $k_3 = (-3.0, 0.7)$, $k_4 = (-0.96, 4.56)$. We take $\omega = vk$ with $v = 1$. Figure 3(b) depicts the fan-out in the directions and magnitudes of the drift velocities for nine different values of diffusivity. Each arrow is U_{s} for one of the

following values of D: $D = 0.1$ (leftmost arrow), $0.2 \ldots 0.9$ (rightmost arrow). For larger values of D, the direction of U approaches more closely that of k_1.

The fan-out effect shown in Figure 3 is due to the different rates at which the contribution from waves decreases as the diffusivity is increased, destroying the exact cancellation imposed at $D = 0$. More light is shed by considering the small-diffusivity and large-diffusivity limits of (19).

1. If $D|k_\ell|^2/\omega_\ell \ll 1 \quad \forall l$ then

$$U_{\mathrm{s}} = -\tfrac{1}{2}\epsilon^2 D^2 \sum_{\ell=1}^{n} \left[A_\ell^2 \frac{|k_\ell|^6}{\omega_\ell^3} k_\ell + \ldots \right]. \tag{23}$$

2. If $D|k_\ell|^2/\omega_\ell \gg 1 \quad \forall l$ then

$$U_{\mathrm{s}} = \tfrac{1}{2}\frac{\epsilon^2}{D^2} \sum_{\ell=1}^{n} \left[A_\ell^2 \frac{\omega_\ell}{|k_\ell|^2} k_\ell + \ldots \right]. \tag{24}$$

In the limit of small diffusivity (1) the drift velocity is proportional to D^2 and the direction is approximately opposite to that of the wave with the largest value of $A^2|k|^6\omega^{-3}$. In the opposite limit (2) the drift velocity is proportional to D^{-2} and the direction is approximately parallel to that of the wave with the largest value of $A^2\omega|k|^{-2}$.

In summary, we derive a general expression for the drift velocity of diffusing particles from a stochastic asymptotic expansion scheme for motion that is Brownian plus a small deterministic forcing. The drift velocity is in general non-zero even when the classical Stokes' drift vanishes. For example, several counterpropagating sinusoidal forcings produce a drift velocity that depends on the diffusion coefficient and the intensities, frequencies and wavenumbers of the forcings. Thus, given a collection of particles with different diffusivities, the deterministic forcings can be tuned to separate particles of a particular type by optimizing their stochastic Stokes' drift.

References

[1] G. G. Stokes, Trans. Camb. Phil. Soc. **8** 441-455 (1847).

[2] O. M. Phillips *The dynamics of the upper ocean* (Cambridge University Press, Cambridge, 1977).

[3] James Lighthill *Waves in Fluids* (Cambridge University Press, Cambridge, 1978).

[4] K. Herterich and K. Hasselmann, J. Phys. Oceanog. **12** 704-711 (1982).

[5] O.N. Mesquita, S. Kane and J.P. Gollub, Phys. Rev. A **45** 3700-3705 (1992).

[6] M. Büttiker, Z. Phys. B **68** 161-167 (1987).

[7] Rolf Landauer, J. Stat. Phys. **53** 233-248 (1988).

[8] Marcelo O. Magnasco, Phys. Rev. Lett. **71** 1477-1480 (1993).

[9] R. Dean Astumian and Martin Bier, Phys. Rev. Lett. **72** 1766-1769 (1994).

[10] C. R. Doering, Il Nuovo Cimento **17** 685–697 (1995).

[11] Juliette Rousselet, Laurence Salome, Armand Ajdari and Jacques Prost, Nature **370** 446-448 (1994).

[12] Mark M. Millonas and Mark I. Dykman, Phys. Lett. A **185** 65-69 (1994).

[13] T.C. Elston and C.R. Doering, J. Stat. Phys. **83** 359-383 (1996).

[14] W. Gardiner *Handbook of stochastic methods* (Springer, Berlin, 1990).

[15] Kalvis Jansons and Grant Lythe, J. Stat. Phys. **90** 227-251 (1998).

Figure 1: *Stochastic Stokes' drift in one dimension.* There is a non-zero drift velocity due to diffusivity even though the classical Stokes' drift vanishes, due to the different rates at which the contributions from each wave decrease as the diffusivity is increased. The drift velocity, (22), is given as a function of diffusivity for $\epsilon = 0.1$, $A_1 = k_1 = \omega_1 = 1$, $k_2 = -2.42$, $\omega_2 = 0.47$.

Figure 2: *Stochastic Stokes' drift: comparison with numerical solution for sinusoidal forcing in one dimension.* (a) Mean value of $\mathbf{X}_t$. (b) Difference between the numerical mean value and the second-order result (22). (c) Distribution of $\mathbf{X}_t$ at $t = 1000$, $\sigma = 0.5$.

Figure 3: *Stochastic Stokes' drift in two dimensions.* (a) Directions and magnitudes of the four sinusoidal forcings. The classical Stokes' drift (18) is zero. (b) Resulting stochastic Stokes' drift as a function of diffusivity. Here $D = 0.1, \ldots, 0.9$ (largest D on the right). In the axis labels, the subscripts indicate vector components. Note the fan-out of more than 180 degrees.